

Distribution of return intervals of extreme events

C. Pennetta^a

Dipartimento di Ingegneria dell'Innovazione, Università di Lecce and CNR-INFM, National Nanotechnology Laboratory, via Arnesano, 73100, Lecce, Italy

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Abstract. The distribution of return intervals of extreme events is studied in time series characterized by finite-term correlations with non-exponential decay. Precisely, it has been analyzed the statistics of the return intervals of extreme values of the resistance fluctuations displayed by resistors with granular structure in nonequilibrium stationary states. The resistance fluctuations are calculated by Monte Carlo simulations using a resistor network approach. It has been found that for highly disordered networks, when the auto-correlation function displays a non-exponential and non-power-law decay, the distribution of return intervals of the extreme values is a stretched exponential, with exponent independent of the threshold.

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1 Introduction

Fluctuations of prices in financial markets, wind speed data or daily precipitations in a given place for the same time windows are typically described by time series $x(t)$ made by uncorrelated records [1, 2]. In these cases, by considering the return intervals r_q of extreme events associated with the overcoming of a given threshold q (i.e. the time intervals between two consecutive occurrences of the condition $x(t) > q$), it can be seen that the r_q are exponentially distributed [1, 3, 4]. In other terms, the probability density function (PDF) of the distribution of the r_q is given by [1, 3, 4]:

$$P_q(r) = (1/R_q) \exp(-r/R_q) \quad (1)$$

where R_q is the mean return interval. Of course, the higher the threshold q , the bigger is the value of R_q . On the other hand, in the last years it has become clear that several other important examples of time series display long-term correlations [2, 5, 6]. This is the case of physiological data (heartbeats [7, 8] and neuron spikes [9]), hydro-meteorological records (daily temperatures [2, 5, 10]), geophysical or astrophysical data (occurrence of earthquakes [11, 12] or solar flares [13]), internet traffic [5] and stock market volatility [2, 14] records. Long-term correlated series are characterized by an auto-correlation function, $C_x(s)$, decaying as a power-law:

$$C_x(s) = \langle x_i x_{i+s} \rangle = \frac{1}{N-s} \sum_{i=1}^{N-s} x_i x_{i+s} \sim s^{-\gamma_x} \quad (2)$$

with exponent γ_x (correlation exponent) between 0 and 1. In this case, the mean correlation time τ , given by the integral over s of $C_x(s)$, diverges. The effect of long-term correlations on the statistics of the r_q has been first studied by Bunde et al. [5] and the conclusions of this study were the following. i) The mean return interval R_q is left unchanged by the presence of long-term correlations; ii) the distribution of return intervals becomes a stretched exponential:

$$P_q(r) = a \exp[-(b r/R_q)^\gamma] \quad (3)$$

where the two exponents γ and γ_x were found to be equal; iii) the return intervals themselves are long-term correlated with a correlation exponent $\gamma' \approx \gamma_x$. As noted by Altmann and Kantz in a very recent paper [2], the result i), obtained by Bunde et al. [5] by statistical arguments, can be identified with Kac's lemma [15] introduced in the context of dynamical systems. The results ii) and iii) were obtained in reference [5] on the ground of numerical calculations performed on long-term correlated and Gaussian time series generated by the algorithm described in reference [16]. It must be noted that the result ii) only applies to linear time series (i.e. to series whose properties are completely defined by the power spectrum and by the probability distribution, regardless of the Fourier phases) [2, 6]. Apart from this restriction, the stretched exponential distribution of the r_q seems to be a general feature in presence of long-term correlations in a time series [2, 6]. It must be underlined that this fact has important consequences on the observation of extreme events: indeed it implies a strong enhancement of the probability of having return intervals well below and well above R_q , in comparison with

^a e-mail: cecilia.pennetta@unile.it

the occurrence of extreme events in an uncorrelated time series. Furthermore, it must be noted that the stretched exponential distribution depends on the parameter γ only, being a and b in equation (3) functions of γ , as shown in reference [2].

In this paper it will be analyzed the effect on the distribution of the r_q of the presence of finite-term correlations with non-exponential decay, a situation which can occur in systems which are approaching criticality, where intermediate behaviors, consisting in a non-exponential and non-power-law decay of correlations can emerge [4]. Precisely, in the following section it will be studied the statistics of the return intervals of extreme values of the resistance fluctuations displayed by a resistor with granular structure in a nonequilibrium stationary state [17–21].

2 Method and results

The time series analyzed in this paper consist in the resistance fluctuations of a thin resistor with granular structure, in contact with a thermal bath at temperature T_0 and biased by an external current I . The resistance values are calculated by using the stationary and biased resistor network (SBRN) model [17–22]. This model describes a thin film with granular structure as a two-dimensional resistor network in a stationary state determined by the competition between two stochastic processes, breaking and recovery of the elementary resistors. A broken elementary resistor (with resistance 10^9 higher than the resistance corresponding to a normal elementary resistor) can be associated with a high resistivity region within the conducting material. Both processes are thermally activated and biased by the external current. The resistance of the network and its fluctuations are calculated by Monte Carlo simulations [17, 18, 21]. Within this model, the level of intrinsic disorder in the network (average fraction of broken resistors in the vanishing current limit [23]) is controlled by a characteristic parameter [20]: $\lambda \equiv (E_D - E_R)/k_B T_0$, where E_D and E_R are the activation energies respectively of the breaking and recovery processes. The intrinsic disorder parameter λ ranges between: $\lambda_{min} < \lambda < \lambda_{max}$, where λ_{max} corresponds to an homogeneous resistor (perfect network) and $\lambda = \lambda_{min} \approx 0$ to the maximum level of intrinsic disorder compatible with a stationary state of the network (stationary resistance fluctuations) [18–21, 23].

In addition to this intrinsic disorder, the SBRN model considers also the presence of a disorder driven by the external current I . As a consequence, for a given value of λ , and for a network of given size, nonequilibrium stationary states of the network exist only when $I \leq I_B$ (breakdown threshold). For contrast, when $I > I_B$ the network undergoes an electrical breakdown associated with an irreversible divergence of its resistance [17–19, 21]. For an arbitrary value of λ this breakdown corresponds to a first order transition from a conducting to a non-conducting state of the network [18–20]. However, at decreasing λ values, when $\lambda \rightarrow \lambda_{min}$, the system becomes more and more close to its critical point [18–20].

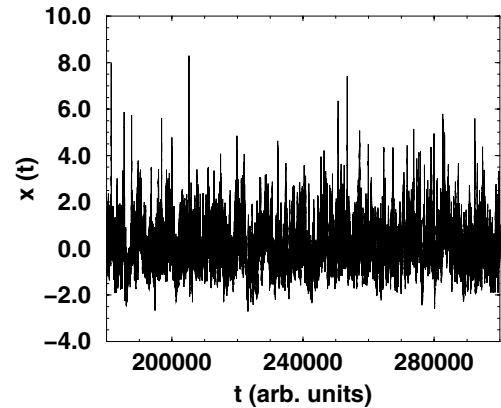


Fig. 1. Normalized resistance fluctuations versus time (this last is expressed in simulation steps). The resistance values have been normalized to provide a zero average and a unit variance for $x(t)$.

All the details about the SBRN model and its results can be found in references [17–21]. However, it must be noted that this model provides a good description of many features associated with nonequilibrium stationary states and with the electrical instability of composites and granular materials [17–19, 24], including the electromigration damage of metallic lines [21]. Finally, it must be underlined that, apart from the specific system described by the SBRN model, the method adopted here for generating the time series can be also viewed as a pure numerical algorithm for generating time series with different and tunable correlation properties.

Then, by indicating with R the resistance of the network (expressed in Ω) and with t the time (expressed in iteration steps), long $R(t)$ time series (typically made of $1 \div 2 \times 10^6$ records) have been generated and analyzed for different values of λ , of the external current and of the network size. Precisely, normalized series with zero average and unit variance have been considered: $x(t) \equiv (R(t) - \langle R \rangle)/\sigma$, where $\langle R \rangle$ is the average value of the network resistance and σ the root-mean-square deviation from the average. The analysis has been performed by calculating the auto-correlation function and the PDF of the x records, the return intervals r_q of the extreme values for different threshold q and their distribution $P_q(r)$ (the values of q are expressed in units of σ). For small λ values (high level of intrinsic disorder), it has been found that C_x displays a non-exponential and non-power-law decay. This behavior is different from that obtained for high λ values (low level of intrinsic disorder), where C_x decays exponentially (consistently with the Lorentzian power spectrum reported in references [17, 18, 20]). By focusing on this situation, of interest for systems which are approaching criticality, in the following of this section results will be shown concerning a network of size 125×125 , biased by a current $I = I_B = 0.011$ A and obtained by taking $\lambda = 0.33$.

The $x(t)$ time series is shown in Figure 1 (only a small portion of the total number of records, $N = 1.6 \times 10^6$, is reported in this figure). As evident from Figure 1, the resistance fluctuations exhibit a strong non-Gaussianity

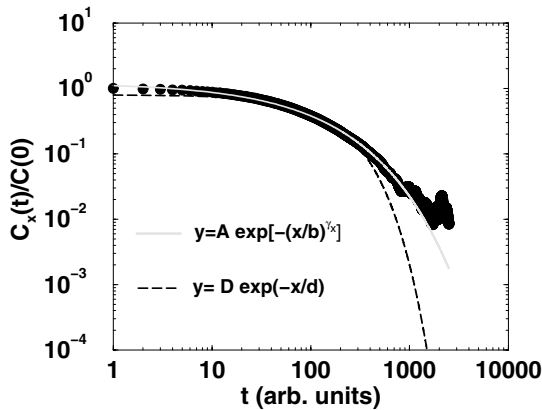


Fig. 2. Plot on a log-log scale of the auto-correlation function of the time series in Figure 1 (black-thick curve). The solid-grey and the black-dashed curves represent the best-fit with a stretched exponential and an exponential, respectively.

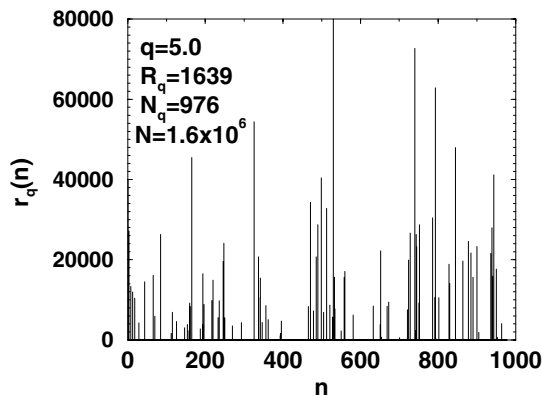


Fig. 3. Return intervals of extreme values above the threshold $q = 5$ (in units of root-mean square deviation) for the time series of Figure 1.

and actually their PDF is well described by the Bramwell-Holdworth-Pinton distribution [25], as discussed in references [19,20]. The auto-correlation function of $x(t)$ is reported in Figure 2. The function significantly deviates from a single exponential and from a power-law while it is well fitted by a stretched exponential:

$$C_x(s) = A \exp\left[-(s/b)^\gamma\right] \quad (4)$$

with the following values of the fitting parameters: $A = 1.23$, $b = 74.9$ and $\gamma = 0.54$. Many other functions have also been considered for the best-fit of the C_x data. However, it has been found that the stretched exponential optimizes the best-fit procedure with the minimum numbers of fitting parameters. It must be remarked that a stretched exponential describes a behavior intermediate between a simple exponential decay (which is obtained for $\gamma = 1$) and a constant behavior (a limit of power-law) for $\gamma \rightarrow 0$. Moreover, the correlation time corresponding to the expression (4) of C_x is finite.

The sequence of the N_q return intervals of the values above the threshold $q = 5$ is plotted in Figure 3 as a sequence of N_q impulses. The figure shows that a succession of very short return intervals ($r_q \ll 100$, the apparently

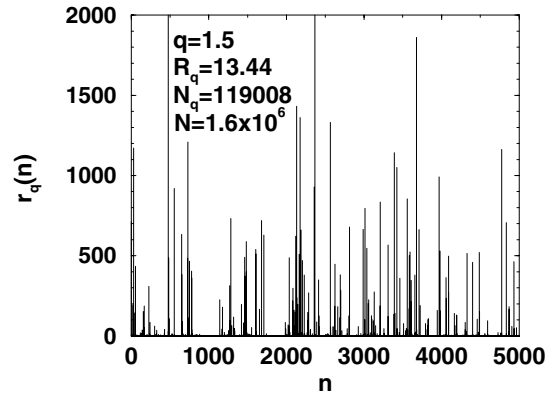


Fig. 4. Return intervals of extreme values above the threshold $q = 1.5$ for the same time series of the previous figures. Here only the first 5000 intervals are shown.

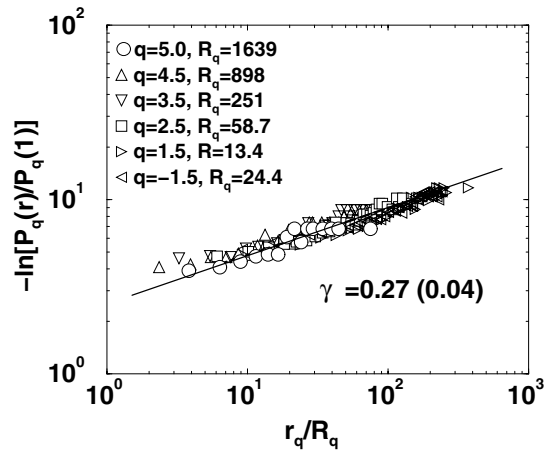


Fig. 5. Double-logarithmic plot of the normalized probability density of the return intervals for different thresholds q .

empty portions of the horizontal axis) is followed by a succession of long intervals, indicating a strong clustering of the extreme events, a feature similar to that exhibited by the data of Bunde et al. [6] which instead concern long-term correlated records and very different from what observed for uncorrelated time series [26]. Thus, the clustering of the extreme events is present even if the x records are not long-term correlated while they are characterized by a finite correlation time. This clustering of the extreme values persists also by lowering the threshold. This is shown in Figure 4 which reports the sequence of the return intervals obtained for $q = 1.5$.

Figure 5 displays the probability density function $P_q(r)$ of the distribution of the return intervals as a function of r_q/R_q for different thresholds q ranging from -1.5 to 5.0 (the probability density has been normalized to $P_q(1)$). A double-logarithmic plot of P_q has been adopted because in this representation a stretched exponential function with exponent γ appears as a straight line of slope γ . Therefore Figure 5 shows that the distribution of the return intervals of extreme values of the x -series is well described by a stretched exponential and that the value of the exponent γ is independent of the threshold q

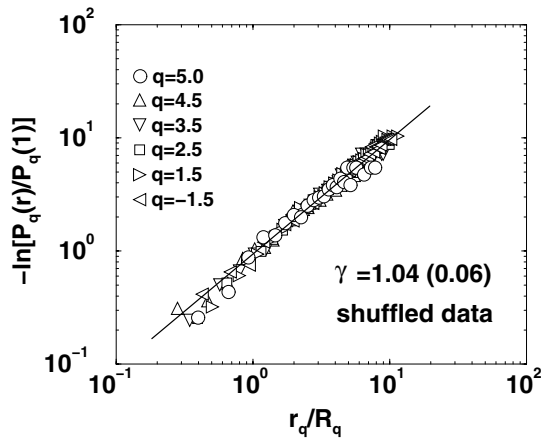


Fig. 6. The same as Figure 5 but calculated after random shuffling the x -records.

in a large range of q -values. This occurs even in absence of long-term correlations and in presence of a finite correlation time. This not-obvious result agrees with the conclusions of Altmann and Kantz formulated in their recent paper [2]. For comparison, Figure 6 reports the probability density function $P_q(r)$ of the distribution of the return intervals obtained after random shuffling the records of the same x -series: in this case $\gamma = 1$, i.e. the distribution of the r_q is exponential, as it must be for uncorrelated time series.

3 Conclusions

The distribution of return intervals of extreme events has been studied in time series with finite-term correlations. Precisely, it has been analyzed the distribution of return intervals of extreme values of the resistance fluctuations displayed by a resistor with granular structure in nonequilibrium stationary states. The resistance fluctuations were calculated by using the SBRN model based on a resistor network approach [17–21]. It has been found that for highly disordered networks, when the auto-correlation function displays a non-exponential and a non-power-law decay, the distribution of the r_q is well described by a stretched exponential with exponent γ largely independent of the threshold q . This result shows that the stretched exponential distribution describes the distribution of the return intervals of extreme events not only when long-term correlations are present in the time series [2,5,6], but also when finite-term correlations exist among the records, characterized by a non-exponential decay, a situation typical of systems which are approaching criticality.

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